An Algorithm to Estimate a Nonuniform Convergence Bound in the Central Limit Theorem

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Abstract

A nonuniform version of the Berry-Esseen bound was proved. The most important feature of the new bound is a monotonically decreasing function $C(|t|)$ instead of the universal constant $C = 29.1174$: $C(|t|) < C$ if $|t| \geq 3.2$ and $C(|t|) \to 1 + e$ if $|t| \to \infty$ where t is a coordinate of the point.

The function $C(|t|)$ has very complex analytical expression based on indicator functions. A general algorithm was developed in order to estimate values of $C(|t|)$ for an arbitrary t.

1 Introduction

Much of the importance of the central limit theorem follows from its proven adaptability and utility in many areas of mathematics, probability theory and statistics [4]. The origin of the term "central" is not clear. Some authors allude to the central role of the theorem in probability theory, while others refer to the fact that the theorem concerns a measure of central tendency, namely the mean of a normalized sum of random variables.

Let $X, X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with

$$\mathbf{E}X = 0, \mathbf{E}X^2 = 1, \mathbf{E}|X|^3 = \rho < \infty. \tag{1.1}$$

The classical Berry-Esseen Theorem ([2] and [3]) states that

$$H_n(t) = \frac{\sqrt{n}}{\rho}|F_n(t) - \Phi(t)| \leq A \quad \forall n \geq 1 \tag{1.2}$$

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where A is an absolute constant, F_n is the distribution function of $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$, and Φ is a standard normal distribution function.

[9] and [10] established a nonuniform structure of the Berry-Esseen bound:

$$H_n(t) \leq \frac{C}{1+|t|^3} \quad \forall n \geq 1. \tag{1.3}$$

[11] and [12] demonstrated that in fact the constant C in the upper bound (1.3) may be replaced by a decreasing function of $|t| \geq 1$:

$$H_n(t) \leq C(|t|) \cdot |t|^{-3} \quad \forall n \geq 1. \tag{1.4}$$

The best known values of the above constants are as follow

$$A \leq 0.7655; \tag{1.5}$$

$$C \leq 29.1174 \approx 0.7655 \cdot \left(1 + \left(\frac{10}{3}\right)^3\right) \tag{1.6}$$

where constants (1.5) and (1.6) were proved by [13] and [8].

2 Main Results

In this section we formulate the algorithm in order to estimate values of the function $C(|t|)$ for an arbitrary value of the argument. Without loss of generality we consider the case $t > 0$ only, because the estimation procedure is symmetrical against the point of origin.

Theorem 2.1 *Under conditions of the Berry-Esseen Theorem (1.1) Tables 1 and 2 represent values for the upper bound*

$$\sup_{x \geq t}\{\frac{\sqrt{n}x^3}{\rho}|F_n(x) - \Phi(x)|\} \leq C(t) = \max\{B_T(t), B_C(t)\}$$

where functions in the right part of the above equation are defined in (3.7) and (3.33a - 3.33c).

Very briefly, the proof structure of the Theorem 2.1 may be formulated as follows.

The target is to estimate $\sup\limits_{t \geq t_0}\{t^3 H_n(t)\}$, and we can exploit exponential rate of decline to zero of the standard normal distribution function $1 - \Phi(t)$ if $t \to \infty$. Respectively, we will split the task into 2 parts:

A1) `tail`: $[\psi(n,t_0),\infty[$, see Lemma 3;

A2) `center`: $[t_0,\psi(n,t_0)]$, see Lemma 7

where $\psi(n,t_0)$ is an increasing function of both arguments t_0 and n (to be defined in (3.6)).

Firstly, we construct the upper bound $B_T(t)$ for $\frac{\sqrt{n}}{\rho}t^3(1-F_n(t)), t \geq \psi(n,t_0)$. This bound has an essential property: $B_T(t) \geq \frac{\sqrt{n}}{\rho}t^3(1-\Phi(t))\,\forall t \in [\psi(n,t_0),\infty[$.

The second step is a much more complex. Using truncation method and results of the Lemma 2 we construct the upper bound $B_C(t)$ for $t^3H_n(t), t_0 \leq t \leq \psi(n,t_0)$.

Subject to the special conditions, both bounds $B_T(t)$ and $B_C(t)$ represent decreasing functions of t and independent on n.

Finally, the required value $C(t_0)$ will be computed as a maximum of the "tail" and "center" bounds.

The Table 1 demonstrates 1) improvement of the nonuniform bound (1.3) with the constant (1.6) if $|t| \geq 3.2$; 2) improvement of the uniform bound (1.2) with the constant (1.5) if $|t| \geq 3.3$.

All values in the Tables 1 and 2 were computed using the following Algorithm where $\tau(t)$ and $b(t)$ are an important components of the truncation parameter $h(t)$ to be defined in (3.6) and (3.14).

Algorithm 1. (for computation of the values of the upper bound $C(t)$ in the Tables 1 and 2)

1: Enter value of the argument $t \geq 3.18$;

2: compute

$$\overline{\tau}(t) := \min\{0.5(1+\sqrt{1-\frac{10}{t^2}}), 1-\frac{\sqrt{3}}{t}\}$$

(this step follows from conditions (3.34d) and (3.34e));

3: compute

$$\overline{b}(t) := \sqrt[3]{\frac{30}{1+e}}$$

(this step follows from the structure of the "tail" bound (3.7));

4: compute

$$\underline{\tau}(t) := \max\{\tau_1, 0.5(1-\sqrt{1-\frac{10}{t^2}})\}$$

Table 1: Values of the upper bound $C(t)$ where τ and b are components of the truncation parameter h to be defined in the Section 3.

t	τ	b	C(t)	Bound: (1.4)	Bound: (1.3)
3.18	0.4553	1.9690	28.4057	0.88333358	0.87823442
3.19	0.4570	1.9670	28.3187	0.87237015	0.87024709
3.20	0.4587	1.9650	28.2363	**0.8617025**	0.86235485
3.21	0.4604	1.9637	28.1563	0.85125797	0.85455633
3.22	0.4601	1.9617	28.0809	0.84109132	0.84685015
3.23	0.4588	1.9597	28.0052	0.83105872	0.83923498
3.24	0.4584	1.9577	27.9293	0.82115399	0.83170951
3.25	0.4581	1.9557	27.8532	0.81138124	0.82427245
3.26	0.4577	1.9547	27.7743	0.80166121	0.81692251
3.27	0.4563	1.9527	27.6980	0.79214601	0.80965846
3.28	0.4559	1.9507	27.6215	0.78275383	0.80247906
3.29	0.4555	1.9487	27.5448	0.77348593	0.79538311
3.30	0.4551	1.9467	27.4681	**0.7643403**	0.78836942
3.40	0.4506	1.9284	26.6933	0.67915056	0.72250902
3.50	0.4461	1.9097	25.9186	0.60451529	0.66370384
3.60	0.4439	1.8907	25.1491	0.53903300	0.61104599
3.70	0.4419	1.8717	24.3886	0.48148354	0.56376222
3.80	0.4402	1.8517	23.6406	0.43083173	0.52119151
3.90	0.4399	1.8327	22.9052	0.38613551	0.48276685
4.00	0.4400	1.8137	22.1853	0.34664525	0.44800024
4.10	0.4405	1.7939	21.4825	0.31169807	0.41647019
4.20	0.4406	1.7749	20.7969	0.28070572	0.38781182
4.30	0.4412	1.7559	20.1302	0.25318851	0.36170787
4.40	0.4434	1.7359	19.4829	0.22871601	0.33788194
4.50	0.4441	1.7179	18.8552	0.20691545	0.31609246
4.60	0.4465	1.6989	18.2478	0.18747242	0.29612776
4.70	0.4485	1.6809	17.6609	0.17010635	0.27780183
4.80	0.4502	1.6619	17.0947	0.15457488	0.26095081
4.90	0.4525	1.6449	16.5490	0.14066430	0.24543000
5.00	0.4556	1.6269	16.0240	0.12819173	0.23111131

where
$$\tau_1 = \frac{1+\sqrt{1-4p}}{2} \quad \text{if} \quad 4p \le 1, p = \frac{2(b(t)-1)}{b^2(t)} - \frac{1}{t^2},$$
$$\text{alternatively,} \quad \tau_1 = -\infty$$
(this step follows from conditions (3.34b) and (3.34d));

5: compute
$$\underline{b}(t) := \frac{2t}{t+\sqrt{t^2-6}}$$
(this step follows from condition (3.34c));

6: enter parameters $\tau(t)$ and $b(t)$ within the ranges: $\underline{\tau}(t) \le \tau(t) \le \overline{\tau}(t)$ and $\underline{b}(t) \le b(t) \le \overline{b}(t)$;

7: check conditions (3.31), (3.37) and (3.38);

8: compute upper bounds for the parameters $\gamma(t), \mu(t)$ and $\beta(t)$ according to (3.13), (3.48) and (3.53);

9: check conditions (3.50) and (3.52a - 3.52c);

10: compute low and upper bounds $\underline{m}_2(t)$ and $\overline{m}_2(t)$ for $m_2(t)$ according to (3.54) using the "center" condition (3.13);

11: compute low bound for $\delta(t)$ according to (3.47) and $\alpha_k(t), k = 0..3, \Delta(t)$ according to (3.17) and (3.18);

12: compute $\eta(t) := |\mu(t)|\sqrt{\overline{m}_2(t)}(3\sqrt{\overline{m}_2(t)} + |\mu(t)|\sqrt{\beta(t)})$ using upper bounds (3.13), (3.48) and (3.54).

13: Finally, $C(t) = \max\{B_T(t), B_C(t)\}$ where the bound $B_T(t)$ is defined in (3.7), and the bound $B_C(t)$ is defined in (3.33a - 3.33c).

Remark 1 *The pair of parameters $\left(\underline{\tau}(t), \overline{b}(t)\right)$ will pass all required conditions if $t \ge 3.18$. In order to optimize selection of the parameters we can consider all possible values from the intervals $[\underline{\tau}(t), \overline{\tau}(t)]$ and $\left[\underline{b}(t), \overline{b}(t)\right]$ with ordered steps. A Pentium 4, 2.8GHz, 512MB RAM, computer was used for the computations which were conducted according to the special program written in C. The total computation time for all values in the Tables 1 and 2 with both steps equal to* 0.001 *was less than 1 min.*

Table 2: Further values of the upper bound $C(t)$.

t	τ	b	C(t)	Bound: (1.4)	Bound: (1.3)
6.00	0.4843	1.4696	11.8046	0.05465073	0.13419355
7.00	0.5166	1.3450	9.0590	0.02641108	0.08465116
8.00	0.5475	1.2486	7.2512	0.01416244	0.05676413
9.00	0.5765	1.1749	6.0329	0.00827556	0.03989041
10.00	0.6298	1.1555	5.7370	0.00573698	0.02909091
11.00	0.6625	1.1461	5.5971	0.00420522	0.02186186
12.00	0.6867	1.1381	5.4808	0.00317173	0.01684211
13.00	0.7078	1.1311	5.3802	0.00244890	0.01324841
14.00	0.7253	1.1251	5.2951	0.00192969	0.01060838
15.00	0.7405	1.1191	5.2108	0.00154394	0.00862559
16.00	0.7537	1.1141	5.1413	0.00125519	0.00710764
17.00	0.7661	1.1091	5.0724	0.00103244	0.00592593
18.00	0.7768	1.1051	5.0177	0.00086037	0.00499229
19.00	0.7868	1.1011	4.9634	0.00072363	0.00424490
20.00	0.7954	1.0971	4.9095	0.00061369	0.00363955
30.00	0.8543	1.0709	4.5661	0.00016911	0.00107848
40.00	0.8857	1.0568	4.3888	0.00006858	0.00045499
50.00	0.9054	1.0475	4.2732	0.00003419	0.00023296
60.00	0.9191	1.0400	4.1827	0.00001936	0.00013481
70.00	0.9293	1.0351	4.1237	0.00001202	0.00008490
80.00	0.9373	1.0318	4.0843	0.00000798	0.00005687
90.00	0.9428	1.0291	4.0527	0.00000556	0.00003995
100.00	0.9477	1.0263	4.0200	0.00000402	0.00002912
∞	1-	1+	**3.7183**	0+	0+

3 Proofs

The proposed method is based on the following truncation

$$Y := \begin{cases} X & \text{if } \ |X| \le h \\ 0 & \text{otherwise} \end{cases}$$

where $h > 0$ is a truncation parameter, and may be regarded as an extension of [7] and [8].

We will denote by F and Q distribution functions of random variables X and Y.

Lemma 1 *(Markov Inequality) Suppose that ℓ is an arbitrary non-decreasing and non-negative function. Then, for any $h > 0$:*

$$\mathbb{P}(|X| \ge h) \le \frac{\mathbf{E}\ell(|X|)}{\ell(h)}, \ell(h) > 0. \tag{3.1}$$

Lemma 2 *(Truncation) The following upper bounds are valid for an arbitrary parameters $s \ge 0$ and $h > 0$:*

$$\beta := \mathbf{E}\exp\{sY\} \le 1 + \frac{s^2}{2} + \frac{\rho}{h^3}(\exp\{sh\} - 1); \tag{3.2a}$$

$$m_1 := \mathbf{E}Y\exp\{sY\} \le s + \frac{\rho}{h^2}\exp\{sh\}; \tag{3.2b}$$

$$m_2 := \mathbf{E}Y^2\exp\{sY\} \le 1 + \frac{\rho}{h}\exp\{sh\}; \tag{3.2c}$$

$$m_3 := \mathbf{E}|Y|^3\exp\{sY\} \le \rho\exp\{sh\}; \tag{3.2d}$$

$$\mathbf{E}|Y|\exp\{sY\} \le \sqrt{\beta m_2}. \tag{3.2e}$$

Proof: Proofs of (3.2a), (3.2b), (3.2c) and (3.2d) are similar and based on the Taylor representation

$$\exp\{sY\} = \sum_{i=0}^{\infty} \frac{(sY)^i}{i!}. \tag{3.3}$$

The following relations are valid according to (3.1)

$$\mathbf{E}Y \le \mathbf{E}X + \int_{|X|\ge h} |X|F(dX) \le \frac{\rho}{h^2}; \tag{3.4a}$$

$$\mathbf{E}Y^2 \le \mathbf{E}X^2 = 1 \le 1 + \frac{\rho}{h}; \tag{3.4b}$$

$$\mathbf{E}Y^i \le \mathbf{E}|X|^3 h^{i-3} \le \rho h^{i-3}, i \ge 3. \tag{3.4c}$$

Combining (3.3) with (3.4a), (3.4b) and (3.4c) we will obtain the bounds (3.2a-3.2d).

The proof of (3.2e) is based on the definitions (3.2a) and (3.2c) and follows from Hölder's inequality. ■

In order to simplify notations we will omit dependence between parameters τ, b, h, c, s and the coordinate of the point t.

Lemma 3 *(`Tail approximation`) Suppose that*

$$\frac{t\sqrt{n}}{h} - \frac{sn}{2h} - c \geq 0 \quad or \quad t^2 \geq \varphi_{n,t}(a,b,c) \tag{3.5}$$

where $a > 0, t > 1, b > c \geq 1$ *and*

$$\psi(n,t) := \varphi_{n,t}(a,b,c) = \frac{b^2}{2(b-c)} \log \frac{\sqrt{n}t^3}{\rho a}, \; h = \frac{\sqrt{n}t}{b}, \; s = \frac{1}{h} \log \frac{\sqrt{n}t^3}{\rho a} > 0. \tag{3.6}$$

Then,

$$t^3 \cdot H_n(t) \leq B_T(t) = b^3(1+e). \tag{3.7}$$

Proof: According to (3.1),

$$1 - F_n(t) \leq 1 - Q_n(t) + 1 - (1 - \mathbb{P}(|X| > h))^n$$

$$\leq 1 - Q_n(t) + n\mathbb{P}(|X| > h) \leq \beta^n \exp\{-st\sqrt{n}\} + \frac{b^3 \rho}{\sqrt{n}t^3} \tag{3.8}$$

where Q_n is a distribution function of $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j$.

According to (3.2a),

$$\beta = \mathbf{E} \exp\{sY\} \leq 1 + \frac{s^2}{2} + \frac{b^3}{an} \leq \exp\{\frac{s^2}{2} + \frac{b^3}{an}\}. \tag{3.9}$$

The following relations are valid as a consequence of the condition (3.5)

$$1 - Q_n(t) \leq \beta^n \exp\{-st\sqrt{n}\}$$

$$\leq \left[\frac{a\rho}{\sqrt{n}t^3}\right]^c \exp\{-sh\left(\frac{t\sqrt{n}}{h} - \frac{sn}{2h} - c\right) + \frac{b^3}{a}\} \leq \left[\frac{a\rho}{\sqrt{n}t^3}\right]^c \exp\{\frac{b^3}{a}\}. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain

$$1 - F_n(t) \leq \frac{\rho b^3}{\sqrt{n}t^3} + \left[\frac{a\rho}{\sqrt{n}t^3}\right]^c \exp\{\frac{b^3}{a}\}. \tag{3.11}$$

On the other hand,

$$1 - \Phi(t) \leq \frac{1}{t\sqrt{2\pi}} \int_t^{\infty} v \exp\{-\frac{v^2}{2}\} dv = \frac{1}{t\sqrt{2\pi}} \exp\{-\frac{t^2}{2}\} \leq \frac{1}{\sqrt{2\pi}} \left[\frac{a\rho}{\sqrt{n}t^3}\right]^c$$

where the last formula was obtained using condition (3.5), $t \geq 1$, and $\frac{b^2}{4(b-c)} \geq c$ if $b \geq c$.

As far as above estimator of $1 - \Phi(t)$ is smaller comparing with (3.11) we can ignore it:

$$H_n(t) \leq \frac{b^3}{t^3} + \frac{a}{t^3} \left[\frac{a\rho}{\sqrt{n}t^3}\right]^{c-1} \exp\{\frac{b^3}{a}\}. \tag{3.12}$$

Maximizing (3.11) as a function of a we find $a = \frac{b^3}{c}$ and

$$\inf_{a>0}\{a^c \exp\{\frac{b^3}{a}\}\} = \left[\frac{b^3 e}{c}\right]^c.$$

Taking into account that $\frac{a\rho}{\sqrt{n}t^3} < 1$ we conclude that the upper bound of H_n is a decreasing function of c. We obtain required result if $c = 1$. ■

3.1 Center approximation

Suppose that $t_0^2 \le t^2 \le \varphi_{n,t}(a,b,c)$ where $a = \frac{b^3}{c}$ or

$$\frac{\rho}{\sqrt{n}} \le c\left(\frac{t}{b}\right)^3 \exp\{2(c-b)\left(\frac{t}{b}\right)^2\} = \gamma(t). \tag{3.13}$$

It will be more convenient for us to redefine here some of the variables of the Lemma 3:

$$h = \tau\sqrt{n}t,\ r = t(1-\tau),\ s = \frac{r}{\sqrt{n}},\ \varepsilon = \frac{\rho}{\tau^3\sqrt{n}t^3}\exp\{\tau(1-\tau)t^2\},\ 0<\tau<1; \tag{3.14}$$

$$G_n(x) = \beta^{-n}\int_{-\infty}^{x} e^{ru}dQ_n(u).$$

It is easy to verify that

$$G_n(x) = G^{n\star}(\sqrt{n}x),\ G(x) = \beta^{-1}\int_{-\infty}^{x} e^{su}dQ(u).$$

Assuming that random variable Z obeys distribution function G, the following relations are valid

$$\mathbf{E}Z = \mu = \frac{m_1}{\beta},\quad \mathbf{E}(Z-\mu)^2 = \delta^2 = \frac{m_2}{\beta} - \mu^2. \tag{3.15}$$

Besides,

$$1 - Q_n(t) = \beta^n\int_t^{\infty} e^{-ru}dG_n(u),\quad 1-\Phi(t) = \exp\{\frac{r^2}{2}\}\int_t^{\infty} e^{-ru}d\Phi(u-r). \tag{3.16}$$

The following notations will be used below

$$\alpha_k = t_0^{3-k}\tau^{-k}\exp\{-0.5(1-\tau)^2t_0^2\},\quad k = 0..3; \tag{3.17}$$

$$\log\Delta = \frac{c}{\tau^3 b^3}\exp\{t_0^2\left(\tau(1-\tau) + \frac{2(c-b)}{b^2}\right)\}; \tag{3.18}$$

$$\mu = \frac{m_1}{\beta}; \tag{3.19}$$

$$\eta = |\mu|\sqrt{m_2}(3\sqrt{m_2} + |\mu|\sqrt{\beta}). \tag{3.20}$$

Lemma 4 *The following bound is valid*

$$\frac{\sqrt{n}}{\rho}|G_n(x\delta + \sqrt{n}\mu) - \Phi(x)| \leq 0.7655q \tag{3.21}$$

where

$$q = \frac{1}{\beta\rho\delta^3}\int |y-\mu|^3 e^{sy} dQ(y) \leq \sqrt{\frac{\beta}{(\underline{m}_2 - \beta\mu^2)^3}}\left(e^{sh} + \eta\right). \tag{3.22}$$

Proof: Inequality (3.21) follows from (1.2), (1.5) and (3.15). Next, we consider upper bound (3.22)

$$q = \frac{1}{\beta\rho\delta^3}\int |y-\mu|^3 e^{sy} dQ(y) \leq \frac{1}{\beta\rho\delta^3}\int (|y| + |\mu|)(y^2 - 2\mu y + \mu^2) dQ(y)$$

$$\leq \frac{1}{\beta\rho\delta^3}\int \left(|y|^3 + 3|\mu|y^2 + \mu^2|y| - |\mu|^3(2\beta - 1)\right) e^{sy} dQ(y)$$

$$\leq \frac{1}{\beta\rho\delta^3}\int \left(|y|^3 + 3|\mu|y^2 + \mu^2|y|\right) e^{sy} dQ(y). \tag{3.23}$$

The required bound may be deduced as a consequence of (3.15) and (3.23) plus (3.2d) and (3.2e). ■

Lemma 5 *Suppose that*

$$\beta \leq 1 + \frac{\rho s^2 h}{6}. \tag{3.24}$$

Then,

$$|\Phi(x\delta + \sqrt{n}\mu - r) - \Phi(x\delta)| \leq \frac{\rho}{h^2} e^{sh}. \tag{3.25}$$

Proof: Based on the definition of normal distribution we have

$$|\Phi(x\delta + \sqrt{n}\mu - r) - \Phi(x\delta)| \leq \frac{|\sqrt{n}\mu - r|}{\sqrt{2\pi}} = \sqrt{\frac{n}{2\pi}}|\mu - s|. \tag{3.26}$$

As far as $\beta \geq 1$, the following relation is valid according to (3.2b)

$$\mu - s \leq m_1 - s \leq \frac{\rho}{h^2} e^{sh}. \tag{3.27}$$

The inequality

$$m_1 \geq s - \frac{\rho}{h^2}\left(1 + sh + 0.5(sh)^2\right)$$

follows from $x\exp\{sx\} \geq x(1 + sx + 0.5(sx)^2)$ $\forall x \in \mathbb{R}$, and from the low bounds:

$$\mathbf{E}Y \geq -\frac{\rho}{h^2}; \mathbf{E}Y^2 \geq 1 - \frac{\rho}{h}; \mathbf{E}Y^3 \geq -\rho. \tag{3.28}$$

Therefore,

$$s - \mu \le \frac{s - m_1}{\beta} + (\beta - 1)s \le \frac{\rho}{h^2}(1 + sh + 0.5(sh)^2) + (\beta - 1)s \le \frac{\rho}{h^2}e^{sh} \quad (3.29)$$

subject to the condition (3.24). ■

Next, we use the property $\rho \ge 1$, which follows from Hölder's inequality applied to the Berry-Esseen condition $\mathbf{E}X^2 = 1$.

In accordance with (3.2a) and (3.13)

$$\beta \le 1 + \frac{t_0^2(1-\tau)^2}{2n} + \frac{\rho}{(\tau t_0\sqrt{n})^3}\exp\{\tau(1-\tau)t_0^2\}$$

$$\le 1 + \frac{\gamma(t_0)}{2\sqrt{n}}\left(t_0^2(1-\tau)^2 + \frac{2\gamma(t_0)\exp\{\tau(1-\tau)t_0^2\}}{(\tau t_0)^3}\right). \quad (3.30)$$

Combining (3.24) and (3.30) we obtain stronger condition

$$c \cdot \exp\{\frac{2(c-b)t_0^2}{b^2}\}\left(t_0^2 + \frac{2c \cdot \exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\}}{b^3\tau^3(1-\tau)^2}\right) \le \frac{b^3\tau}{3}. \quad (3.31)$$

Lemma 6 *The following upper bound is valid*

$$\sup_{t \ge t_0}|\Phi(t\delta) - \Phi(t)| \le \frac{t_0|\delta - 1|\left[\exp\{-\frac{t_0^2}{2}\} + \exp\{-\frac{(\delta t_0)^2}{2}\}\right]}{2\sqrt{2\pi}}. \quad (3.32)$$

Proof: The required inequality follows from convexity of the exponential function.

Lemma 7 *(`Center approximation`) The following upper bound is valid*

$$|t|^3 H_n(t) \le B_C(t_0) = \tau^{-3} + \alpha_3\Delta \quad (3.33a)$$

$$+\sqrt{\frac{2}{\pi}}\left[\alpha_2 + 0.25 \cdot \alpha_1 t_0\left(\exp\{-\frac{t_0^2}{2}\} + \exp\{-\frac{(\delta t_0)^2}{2}\}\right)\right] \quad (3.33b)$$

$$+1.531\sqrt{\frac{\beta}{(\underline{m}_2 - \beta\mu^2)^3}}\left(\alpha_0 + \eta t_0^3\exp\{-\frac{t_0^2(1-\tau^2)}{2}\}\right); \quad (3.33c)$$

under conditions (3.13), (3.31) and

$$\frac{2(b-c)}{b^2} > \tau(1-\tau); \quad (3.34a)$$

$$t_0^2 \ge \left(\frac{2(b-c)}{b^2} - \tau(1-\tau)\right)^{-1}; \quad (3.34b)$$

$$t_0^2 \ge \frac{3b^2}{2(b-c)}; \quad (3.34c)$$

$$t_0^2 \ge \frac{5}{2\tau(1-\tau)}; \quad (3.34d)$$

$$t_0^2 \ge \frac{3}{(1-\tau)^2} \quad (3.34e)$$

where $0 < \tau < 1$ *and* $b > c \geq 1$.

Proof: Using (3.9) and (3.13) we obtain the upper bound

$$|\beta - 1 - \frac{r^2}{2n}| \leq \frac{\varepsilon}{n} \leq \frac{c}{n\tau^3 b^3} \exp\{t_0^2 \left(\tau(1-\tau) + \frac{2(c-b)}{b^2}\right)\}. \tag{3.35}$$

Suppose that

$$5\varepsilon \leq r^2; \tag{3.36a}$$

$$r^2 \leq 2\varepsilon\sqrt{n}. \tag{3.36b}$$

Inequality (3.36a) follows from stronger condition

$$5 \cdot c \cdot \exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\} \leq \tau^3(1-\tau)^2 b^3 t_0^2 \tag{3.37}$$

under (3.34a): condition that the left part of (3.37) is a non-increasing function of t_0 (means, the inequality will be valid $\forall t \geq t_0$).

Inequality (3.36b) follows from stronger condition

$$(1-\tau)^2\tau^3 t_0^5 \exp\{-\tau(1-\tau)t_0^2\} \leq 2 \tag{3.38}$$

under (3.34d): condition that the left part of (3.38) is a non-increasing function of t_0.

According to (3.2a) $\beta \leq \exp\{\frac{s^2}{2} + \frac{\varepsilon}{n}\}$. Therefore,

$$\frac{\sqrt{n}}{\rho} e^{-rt}|\beta^n - e^{0.5r^2}| \leq \frac{\sqrt{n}}{\rho}\varepsilon \exp\{\frac{r^2}{2} + \varepsilon - rt\} \leq \alpha_3 \Delta t^{-3} \tag{3.39}$$

where $\varepsilon \leq \log \Delta$ as an equivalent of (3.13), and Δ is defined in (3.18).

The inequality (3.40) is similar to (3.8). Then, we use representation (3.16) and bound (3.39)

$$|H_n(t)| \leq \frac{\sqrt{n}}{\rho}|\Phi(t) - Q_n(t)| + \frac{n^{\frac{3}{2}}}{\rho}\mathbb{P}(|X| > h) \tag{3.40}$$

$$\leq \frac{\sqrt{n}}{\rho}\left(|\beta^n - e^{0.5r^2}|\int_t^\infty e^{-ru} dG_n(u) + e^{0.5r^2}|\int_t^\infty e^{-ru} d\left[G_n(u) - \Phi(u-r)\right]|\right) + (\tau t)^{-3}$$

$$\leq \frac{\sqrt{n}}{\rho}\left(|\beta^n - e^{0.5r^2}|e^{-rt} + 2\exp\{\frac{r^2}{2} - rt\} \sup_{x\in\mathbb{R}}|G_n(x) - \Phi(x-r)|\right) + (\tau t)^{-3}$$

$$\leq \left(\alpha_3\Delta + \tau^{-3}\right) t^{-3} + \frac{2\sqrt{n}}{\rho}\exp\{\frac{r^2}{2} - rt\} \sup_{x\in\mathbb{R}}|G_n(x) - \Phi(x-r)|. \tag{3.41}$$

Consider the last term in (3.41)

$$\sup_{x\in\mathbb{R}}|G_n(x) - \Phi(x-r)| \leq \sup_{x\in\mathbb{R}}\{|\Phi(x\delta + \sqrt{n}\mu - r) - \Phi(x\delta)|$$

$$+ |\Phi(x\delta) - \Phi(x)| + |\Phi(x) - G_n(x\delta + \sqrt{n}\mu)|\}. \tag{3.42}$$

Using (3.25) we deduce that

$$\frac{2\sqrt{n}}{\rho} \exp\{0.5r^2 - rt\} |\Phi(x\delta + \sqrt{n}\mu - r) - \Phi(x\delta)| \le \sqrt{\frac{2}{\pi}} \frac{\alpha_2}{t^3} \tag{3.43}$$

under (3.34e): condition which ensure that α_2 is a non-increasing function of t_0.

Based on the general inequality

$$|\delta - 1| \le 0.5 \max\left(\delta^2 - 1, \frac{1-\delta^2}{\delta}\right)$$

and result of the Lemma 6 we can conclude that

$$|\Phi(x\delta) - \Phi(x)| \le \frac{t_0\zeta}{4\sqrt{2\pi}} \left[\exp\{-\frac{t_0^2}{2}\} + \exp\{-\frac{(\delta t_0)^2}{2}\}\right] \tag{3.44}$$

subject to the conditions

$$\max\{0.25, 1 - 0.5\zeta\} \le \delta^2 \le 1 + \zeta, \zeta > 0. \tag{3.45}$$

We have

$$\underline{m}_2 = 1 - \frac{\rho}{h}(1 + sh) \le m_2 \le 1 + \frac{\rho}{h} e^{sh} = \overline{m}_2 \tag{3.46}$$

where left inequality is valid according to $e^{sx} \ge 1 + sx \; \forall x$, and (3.28).

The right inequality is valid according to (3.2c). It follows from (3.46) and

$$\frac{1}{\beta} \ge 2 - \beta, \beta \ge 1,$$

that

$$\delta^2 \ge \frac{1}{\beta} - \frac{\rho}{h}(1 + sh) - \mu^2 \ge 2 - \beta - \frac{\rho}{h}(1 + sh) - \mu^2. \tag{3.47}$$

Furthermore, by (3.27) and (3.29),

$$|\mu| \le t_0 \cdot \gamma(t_0) \left(1 - \tau + \frac{c \cdot \exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\}}{\tau^2 b^3}\right). \tag{3.48}$$

By inserting

$$\zeta = \frac{\rho}{h} e^{sh}$$

into (3.44) we have

$$\frac{2\sqrt{n}}{\rho} \exp\{0.5r^2 - rt\} \cdot |\Phi(x\delta) - \Phi(x)| \tag{3.49a}$$

$$\le \frac{\alpha_1}{2\sqrt{2\pi} \cdot t^3} t_0 \left[\exp\{-\frac{t_0^2}{2}\} + \exp\{-\frac{(\delta t_0)^2}{2}\}\right] \tag{3.49b}$$

under (3.34e): condition which ensure that that α_1 is a non-increasing function of t_0.

The following inequality was derived using (3.13) and (3.47) and corresponds to the condition $\delta^2 \geq 0.25$ of (3.45)

$$\beta - 1 + \mu^2 + c \cdot t_0^2 b^{-3} \exp\{2(c-b)\left(\frac{t_0}{b}\right)^2\}\left[\tau^{-1} + t_0^2(1-\tau)\right] \leq 0.75. \tag{3.50}$$

under (3.34d): this condition (combined with condition (3.34a)) will ensure that the left part of (3.50) is a non-increasing.

Note that we can use weaker condition

$$t_0 \geq \frac{5b^2}{4(b-c)}$$

in order to ensure that the left part of (3.50) is a non-increasing as a function of t_0. But, we used already stronger (according to (3.34a)) condition (3.34d), which is an essential for (3.38). Respectively, we will leave in above and further cases only condition, which is stronger.

Next inequality corresponds to $\delta^2 \geq 1 - 0.5 \cdot \zeta$ of (3.45) and was obtained using (3.47)

$$1 + \frac{\rho}{h}\left(\frac{e^{sh}}{2} - 1 - sh\right) \geq \beta + \mu^2. \tag{3.51}$$

Then, we apply (3.30) and (3.48) to the right side of (3.51)

$$\tau \cdot t_0 \cdot \gamma(t_0)(0.5t_0^2(1-\tau)^2 + c \cdot \frac{\exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\}}{b^3\tau^3} \tag{3.52a}$$

$$+t_0^2\left(1 - \tau + c \cdot \frac{\exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\}}{b^3\tau^2}\right)^2) \tag{3.52b}$$

$$\leq 0.5 \cdot \exp\{t_0^2\tau(1-\tau)\} - 1 - t_0^2\tau(1-\tau) \tag{3.52c}$$

under condition (3.34d), which ensure that (3.52c) is a non-decreasing; plus (3.34c): condition that the left part represented by (3.52a) and (3.52b) is a non-increasing. The condition $\delta^2 \leq 1 + \zeta$ of (3.44) is always valid according to (3.2c).

We can re-write estimators (3.30) and (3.46) in a more detailed form using "center" condition (3.13)

$$\beta \leq 1 + \gamma^2(t_0)\left(\frac{(1-\tau)^2 t_0^2}{2} + \frac{c \cdot \exp\{t_0^2(\tau(1-\tau) + \frac{2(c-b)}{b^2})\}}{\tau^3 b^3}\right) \tag{3.53}$$

under (3.34a) and (3.34d): conditions, which ensure that the upper estimator (3.53) is a non-increasing;

$$\underline{m}_2 = 1 - \frac{\gamma(t_0)}{t_0}(\frac{1}{\tau} + (1-\tau)t_0^2) \leq m_2 \leq 1 + \frac{\gamma(t_0)\exp\{t_0^2\tau(1-\tau)\}}{t_0\tau} = \overline{m}_2 \tag{3.54}$$

under (3.34b): condition that the upper bound (3.54) is a non-increasing, and under (3.34d): condition that the low bound (3.54) is a non-decreasing.

By (3.21) and (3.22)

$$\frac{2\sqrt{n}}{\rho}\exp\{0.5r^2 - rt\}|G_n(x\delta + \sqrt{n}\mu) - \Phi(x)|$$
$$\leq \frac{1.531}{t^3}\sqrt{\frac{\beta}{(\underline{m}_2 - \beta\mu^2)^3}}\left(\alpha_0 + \eta t_0^3 \exp\{-0.5t_0^2(1-\tau^2)\}\right) \tag{3.55}$$

under (3.34e): condition that α_0 is a non-increasing.

Assuming that $c = 1$, and combining (3.41), (3.42), (3.43), (3.49) and (3.55) under conditions (3.34a) - (3.34e), (3.37), (3.38), (3.31), (3.50) and (3.52b) we obtain required result. ■

Proposition 1 *Suppose that the sample size n is fixed. Then,*

$$\lim_{t\to\infty} C(t) \leq 1 \tag{3.56}$$

under conditions of the Berry-Esseen Theorem (1.1).

Proof We need to consider (3.11) only if sample size n is fixed and t is large enough.

Clearly, we can construct the functions $c(t)$ and $b(t)$: $1 < c(t) < b(t) \longrightarrow 1$ if $t \to \infty$, under "tail" condition (3.5)

$$b(t)\left(1 - \frac{b(t)}{2t^2}\log\frac{\sqrt{n}t^3}{\rho a}\right) - c(t) \geq 0,$$

such that the upper bound for $t^3 H_n(t)$ obtained from (3.12) will tend to 1 if $t \to \infty$.

4 Concluding Remarks

According to [5], the probability literature contains a large body of very elegant mathematical theory which describes the rate of convergence in the central limit theorem. These results often involve a uniform measure of the rate of convergence. Statisticians are sometimes rather skeptical of such theory, pointing out that it is disjoint from the more practical problems, which they encounter. Frequently, they are only interested in the rate of convergence in isolated points.

For example, using the nonuniform bound (1.4) we can construct in analytical form the upper bound for the confidence interval based on the sample mean as an estimator of the location parameter ([1] and [6]):

$$\mathbb{P}(|\frac{1}{n}\sum_{i=1}^{n} X_i - \theta| > \varepsilon) \leq 2\left(1 - \Phi(\sqrt{n}\varepsilon) + \frac{\rho C(\sqrt{n}\varepsilon)}{n^2\varepsilon^3}\right)$$

where $\sqrt{n}\varepsilon \geq 1$. The Table 1 demonstrates advantage of the bound (1.4) if $\sqrt{n}\varepsilon \geq 3.3$.

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